LOCAL HOMEOMORPHISMS THAT ∗-COMMUTE WITH THE SHIFT

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ABSTRACT. Exel and Renault proved that a sliding block code on a one-sided shift space coming from a progressive block map is a local homeomorphism. We provide a counterexample showing that the converse does not hold. We use this example to generalize the notion of progressive to a property of block maps we call *weakly progressive*, and we prove that a sliding block code coming from a weakly progressive block map is a local homeomorphism. We also introduce the notion of a *regressive* block map and prove that a sliding block code ∗-commutes with the shift map if and only if it comes from a regressive block map. We also prove that a sliding block code is a local homeomorphism and ∗-commutes with the shift map if and only if it is a k-fold covering map defined from a regressive block map.

1. INTRODUCTION

In symbolic dynamics one considers spaces of sequences with entries from a finite alphabet A together with a shift map on the space. There are two versions of this theory, one that considers the space of two-sided infinite sequences $A^{\mathbb{Z}}$ with a shift map σ that is a homeomorphism, and one that considers the space of one-sided infinite sequences $A^{\mathbb{N}}$ with a shift map σ that is a local homeomorphism.

Morphisms between shift spaces are called sliding block codes, and one can prove that any such morphism τ_d comes from a block map $d : A^n \to A$ (see [3, 6, 5]). In [3, Theorem 3.4] Hedlund proved that the morphisms on two-sided infinite sequences are precisely functions of the form $\sigma^k \tau_d$ for some $k \in \mathbb{Z}$. In Section 2 of this paper we characterize morphisms on one-sided infinite sequences as functions of the form τ_d (Theorem 2.10). More specifically, the morphisms on one-sided infinite

sequences are functions $\tau_d : A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined from a block map $d :$ $A^n \to A$, for some $n \in \mathbb{N}$, by $(\tau_d(x))_i = d(x_i \cdots x_{i+n-1})$. While this result has been stated on several occasions in the literature we include a proof for completeness.

The first main topic of this paper is to consider when a sliding block code on the one-sided shift space is a local homeomorphism. To date, there is no known characterization. In [2, Theorem 14.3] Exel and Renault proved that if the block map d that defines τ_d is a *progressive* function (Definition 3.2), then τ_d is a local homeomorphism. At the time, it was not known if the converse is true. In Section 3 we provide a counterexample to the converse (Example 3.6). We use this counterexample as motivation to generalize the idea of a progressive block map and introduce what we call a *weakly progressive* block map (Definition 3.7). We prove that if the block map d is weakly progressive then τ_d is a local homeomorphism (Theorem 3.11). This gives weaker hypothesis under which we can conclude that τ_d is a local homeomorphism. We do not yet know if the converse is true; that is, we do not know if a sliding block code that is a local homeomorphism must come from a weakly progressive block map.

The second main topic of this paper is to examine sliding block codes that $*$-commute with the shift. The concept of two functions $*$-commuting was introduced in [1] and further examined in [2]. Let X be a topological space. Then two functions $S, T : X \to X$ $*$-*commute* if they commute and for all $y, z \in X$ with $S(y) = T(z)$ there exists a unique $x \in X$ such that $T(x) = y$ and $S(x) = z$. In Section 4 we introduce the concept of a *regressive* block map (Definition 4.5) and characterize the sliding block codes that $*$-commute with the shift as those for which the block map d is regressive.

Local homeomorphisms that $*$-commute with the shift have interesting properties that we discuss in Section 5. In particular, we prove that a sliding block code is a local homeomorphism and $*$-commutes with the shift if and only if $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a k-fold covering map coming from a regressive block map (Theorem 5.14).

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Notation and conventions: Throughtout this paper we let A be a finite alphabet. Give A the discrete topology, then A is a compact Hausdorff space. Let A^n denote the words of length n, let $A^* := \bigcup_{n \geq 1} A^n$, and let $A^{\mathbb{N}}$ denote the one-sided infinite sequence space of elements in A. Since A is a compact Hausdorff space, $A^{\mathbb{N}}$ with the product topology is also a compact Hausdorff space by Tychonoff's Theorem.

For $\mu \in A^*$ we define a *cylinder set* $Z(\mu) := \{x \in A^{\mathbb{N}} : x_1 \cdots x_{|\mu|} = \mu\}$. Note that the family $\{Z(\mu) : \mu \in A^*\}$ is a basis for $A^{\mathbb{N}}$ and each $Z(\mu)$ is clopen (and therefore compact). Let $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined by $\sigma(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$ be the shift map.

2. CONTINUOUS FUNCTIONS THAT COMMUTE WITH THE SHIFT

In this section we provide a proof of the fact that a function on the one-sided shift space that is continuous and commutes with the shift is a sliding block code. Our proof is constructive in the sense that given an arbitrary continuous function $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ that commutes with the shift map σ, Lemma 2.5 and Proposition 2.6 demonstrate how to construct a block map that defines ϕ.

Definition 2.1. A *block map* is a function $d : A^n \to A$ for some $n \in \mathbb{N}$. For any block map d we define a function $\tau_d : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by $\tau_d(x)_i = d(x_i \cdots x_{i+n-1})$. We call τ_d a *sliding block code*.

Remark 2.2. *Define $d : A^2 \to A$ by $d(a_1 a_2) = a_2$. Then $\tau_d = \sigma$. Note that τ_d is not uniquely determined by the choice of d. For example, define $d' : A^3 \to A$ by $d'(a_1 a_2 a_3) = a_2$. Then $\tau_{d'} = \sigma = \tau_d$.*

Lemma 2.3. *If $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a sliding block code, then ϕ is continuous and commutes with the shift map σ.*

Proof. Since ϕ is a sliding block code there exists $n \in \mathbb{N}$ and a block map $d : A^n \to A$ such that $\tau_d = \phi$. Let $\mu \in A^*$. If $x = x_1 x_2 x_3 \cdots \in \tau_d^{-1}(Z(\mu))$, then one can show that $x \in Z(x_1 x_2 \cdots x_{k+n-1}) \subseteq \tau_d^{-1}(Z(\mu))$ so $\tau_d^{-1}(Z(\mu))$ is open. Therefore τ_d is continuous. Let $x \in A^{\mathbb{N}}$ and observe

$$\tau_d \sigma(x) = \tau_d(x_2 x_3 x_4 \cdots) = d(x_2 \cdots x_{n+1})d(x_3 \cdots x_{n+2}) \cdots$$
$$= \sigma(d(x_1 \cdots x_n)d(x_2 \cdots x_{n+1})d(x_3 \cdots x_{n+2}) \cdots) = \sigma \tau_d(x).$$

$\square$

Remark 2.4. *For each $\mu \in A^*$, $Z(\mu) = \bigsqcup_{a \in A} Z(\mu a)$. Recall that since $A^{\mathbb{N}}$ is compact, any open set $U \subseteq A^{\mathbb{N}}$ is a finite union of basis elements. Let S be a finite subset of A^*. Then for any open set $U = \bigcup_{\mu \in S} Z(\mu)$, we may extend the lengths of the $\mu \in S$ so that we may find $n \in \mathbb{N}$ and a finite set $T \subseteq A^n$ such that $\bigcup_{\mu \in S} Z(\mu) = \bigsqcup_{\nu \in T} Z(\nu)$. Note that for any $\mu \in S$, $|\mu| \leq n$. Therefore for any $\nu \in T$ there exists $\mu \in S$ such that $Z(\nu) \subseteq Z(\mu)$.*

Lemma 2.5. *If $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a continuous function, then there exists $n \in \mathbb{N}$ such that for each $\lambda \in A^n$ there exists a unique $a \in A$ such that $Z(\lambda) \subseteq \phi^{-1}(Z(a))$.*

Proof. Consider $\{Z(a) \mid a \in A\}$. Notice that the $Z(a)$'s are disjoint clopen sets that cover $A^{\mathbb{N}}$. Define $V_a = \phi^{-1}(Z(a))$. Then the V_a's are also disjoint clopen sets that cover $A^{\mathbb{N}}$. Since $A^{\mathbb{N}}$ is compact and the V_a's are closed, the V_a's are also compact. Therefore each V_a is the union of a finite number of basis elements. That is, $V_a = \bigcup_{\mu \in S_a} Z(\mu)$, where S_a is a finite subset of A^*. By Remark 2.4 there exists $m \in \mathbb{N}$ and a finite set $T_a \subseteq A^m$ such that $V_a = \bigcup_{\mu \in S_a} Z(\mu) = \bigsqcup_{\nu \in T_a} Z(\nu)$. Let $T = \bigcup_{a \in A} T_a$. Then $\bigsqcup_{a \in A} V_a = \bigsqcup_{\nu \in T} Z(\nu)$ where T is a finite subset of A^*. Observe that the $Z(\nu)$'s are disjoint since the V_a's are disjoint and that for each $\nu \in T$ there exists a unique $a \in A$ such that $Z(\nu) \subseteq V_a$. By Remark 2.4 there exists $n \in \mathbb{N}$ and a finite set $R \subseteq A^n$ such that $\bigsqcup_{a \in A} V_a = \bigsqcup_{\nu \in T} Z(\nu) = \bigsqcup_{\lambda \in R} Z(\lambda)$. Note that since $|\nu| \leq n$ for every $\nu \in T$ and the $Z(\nu)'s$ are disjoint, then for every $\lambda \in R$ there exists a unique $\nu \in T$ such that $Z(\lambda) \subseteq Z(\nu)$. Recall that $\bigsqcup_{a \in A} V_a$ is a cover of $A^{\mathbb{N}}$. Therefore $\bigsqcup_{\lambda \in R} Z(\lambda)$ is also a cover of $A^{\mathbb{N}}$, and hence $R = A^n$. Observe that for each $\lambda \in A^n$, there exists a unique $\nu \in T$ and a unique $a \in A$ such that $Z(\lambda) \subseteq Z(\nu) \subseteq V_a = \phi^{-1}(Z(a))$. $\qquad\square$

Proposition 2.6. *If $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a continuous function that commutes with the shift map σ, then ϕ is a sliding block code.*

Proof. Since ϕ is continuous, by Lemma 2.5 there exists $n \in \mathbb{N}$ such that for each $\lambda \in A^n$ there exists a unique $a \in A$ such that $Z(\lambda) \subseteq \phi^{-1}(Z(a))$. Define $d : A^n \to A$ by $d(\lambda) = a$ where a is the unique element in A such that $Z(\lambda) \subseteq \phi^{-1}(Z(a))$. The function d is well defined since the element a is unique. We will now show that $\tau_d = \phi$. Let $k \in \mathbb{N}$ and $x \in A^{\mathbb{N}}$. Notice that $\sigma^k(x) \in Z(x_{k+1} \cdots x_{k+n}) \subseteq \phi^{-1}(Z(a))$ for some $a \in A$. Then $\phi(\sigma^k(x))_1 = a$. Also, we have $\tau_d(\sigma^k(x))_1 = d(\sigma^k(x)_1 \cdots \sigma^k(x)_n) = d(x_{k+1} \cdots x_{k+n}) = a$. Therefore

$$\phi(x)_k = \sigma^k(\phi(x))_1 = \phi(\sigma^k(x))_1 = a = \tau_d(\sigma^k(x))_1 = \sigma^k(\tau_d(x))_1 = \tau_d(x)_k.$$
$$\square$$

Example 2.7. Fix $a \in A$ and define $aaa \cdots = \overline{a} \in A^{\mathbb{N}}$. Consider a constant function $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined by $\phi(y) = \overline{a}$ for all $y \in A^{\mathbb{N}}$. Then ϕ is a continuous function that commutes with σ. Notice that for $b \in A$, $\phi^{-1}(Z(b)) = \emptyset$ unless $b = a$, thus we have $\phi^{-1}(Z(a)) = A$ and $n = 1$. So by Proposition 2.6 $d : A \to A$ is defined by $d(b) = a$ for all $b \in A$ and $\tau_d = \phi$.

Example 2.8. Let $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be defined by $\phi(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$. Observe that for $b \in A$, $\phi^{-1}(Z(b)) = \bigsqcup_{a \in A} Z(ab)$. So for all $x_1 x_2 \in A^2$ we have $Z(x_1 x_2) \subset \phi^{-1}(Z(x_2))$. So by Proposition 2.6 $d : A^2 \to A$ is defined by $d(x_1 x_2) = x_2$.

The following proposition shows us the extent to which the function d of Proposition 2.6 is unique.

Proposition 2.9. *Let $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a continuous function that commutes with the shift map σ. Let n be the smallest natural number with the property that there exists a block map $d : A^n \to A$ such that $\tau_d = \phi$. (By Proposition 2.6 such an n exists.) If $m \in \mathbb{N}$ and there exists a function $d' : A^m \to A$ such that $\tau_{d'} = \phi$, then $m \geq n$ and $d'(x_i \cdots x_{m+i-1}) = d(x_i \cdots x_{n+i-1})$ for all $i \in \mathbb{N}$. In particular, if $m = n$, then $d' = d$.*

Proof. Since $\tau_d = \phi = \tau_{d'}$, we have $m \geq n$ by the minimality of n. Also, for all $i \in \mathbb{N}$ and $x \in A^{\mathbb{N}}$ we have $d(x_i \cdots x_{n+i-1}) = \tau_d(x)_i = \phi(x)_i = \tau_{d'}(x)_i = d'(x_i \cdots x_{m+i-1})$. $\quad\square$

Theorem 2.10. *The function $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is continuous and commutes with the shift map σ if and only if ϕ is a sliding block code.*

Proof. The sufficiency is proven in Proposition 2.6 and the necessity is proven in Lemma 2.3. $\quad\square$

The following example illustrates the importance of the function ϕ being continuous.

Example 2.11. Let $A = \{0, 1\}$ and $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be defined as follows: $\phi(0^{\infty}) = 1^{\infty}$, $\phi(1^{\infty}) = 0^{\infty}$, and ϕ is the identity on all other points. It is clear that ϕ commutes with σ, however it is impossible to find a function d such that $\tau_d = \phi$. If $d : A^n \to A$ could be defined for some $n \in \mathbb{N}$, then for the points where ϕ acts as the identity we must have that $d(0^n) = 0$ and $d(1^n) = 1$. However, defining d in this manner would not work for the points 0^{∞} and 1^{∞}.

3. Local homeomorphisms that commute with the shift

In this section we examine properties on the block map that force the induced sliding block code to be a local homomorphism. Exel and Renault proved that if the block map is progressive, then the induced sliding block code is a local homeomorphism [2, Theorem 14.3]. The converse, however, remained an open problem. In this section we prove the converse is false by providing a counterexample in Example 3.6. Specifically, we describe a sliding block code that is a local homeomorphism such that there does not exist a progressive block map that defines it. We then generalize the idea of a progressive block map by defining a weakly progressive block map (Definition 3.7). In Theorem 3.11 we prove that if the block map is a weakly progressive function, then the induced sliding block code is a local homeomorphism. This

gives weaker hypothesis under which we can conclude that a sliding block code is a local homeomorphism.

Definition 3.1. Let X, Y be topological spaces. A continuous function $f : X \to Y$ is a *local homeomorphism* if for every point $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is open in Y and $f : U \to f(U)$ is a homeomorphism.

Definition 3.2. A block map $d : A^n \to A$ is *progressive* if for each fixed $x_1 \cdots x_{n-1} \in A^{n-1}$, the function $p_d^{x_1 \cdots x_{n-1}} : A \to A$ defined by $p_d^{x_1 \cdots x_{n-1}}(a) = d(x_1 \cdots x_{n-1}a)$ is bijective .

Example 3.3. Define $d : A^2 \to A$. by $d(a_1a_2) = a_2$. Then $\tau_d = \sigma$. Fix $a \in A$, let $a_1, a_2 \in A$ and suppose $p_d^a(a_1) = p_d^a(a_2)$. We have

$$a_1 = d(aa_1) = p_d^a(a_1) = p_d^a(a_2) = d(aa_2) = a_2,$$

so p_d^a is injective. Note that $b = p_d^a(b)$ for all $b \in A$ so p_d^a is surjective. Hence d is progressive.

Remark 3.4. *Recall from Remark 2.2 that for $d' : A^3 \to A$ defined by $d(a_1a_2a_3) = a_2$ we have $\tau_{d'} = \sigma = \tau_d$ for the block map d from Example 3.3. Notice that d' is not progressive, therefore it is important that we consider the smallest natural number n such that the function $d : A^n \to A$ defines τ_d.*

Given an arbitrary sliding block code ϕ we wish to determine if there is a progressive block map that defines it. Let n be the smallest natural number such that a block map $d : A^n \to A$ defines ϕ. Then for any $m > n$ and block map $d' : A^m \to A$ that defines ϕ the function d' is not progressive. We observe this by recalling from Proposition 2.9 that $d'(x_i \cdots x_{m+i-1}) = d(x_i \cdots x_{n+i-1})$. Therefore d' can not be bijective.

In this section, when we consider a block map $d : A^n \to A$ that defines τ_d we assume that n is the smallest natural number such that there exists a function $d : A^n \to A$ that defines τ_d. Proposition 2.9 allows us to do this.

Example 3.5. The constant function $d(b) = a$ for all $b \in A$ from Example 2.7 is not progressive.

The following is an example of a sliding block code that is a local homeomorphism and can not be defined from a progressive block map.

Example 3.6. Let $A = \{0, 1, 2, 3\}$ and define $d : A^2 \to A$ by

$d(00) = 0$	$d(01) = 0$	$d(02) = 1$	$d(03) = 1$
$d(10) = 3$	$d(11) = 3$	$d(12) = 2$	$d(13) = 2$
$d(20) = 2$	$d(21) = 2$	$d(22) = 3$	$d(23) = 3$
$d(30) = 1$	$d(31) = 1$	$d(32) = 0$	$d(33) = 0.$

Observe that it is not possible to define a block map $d' : A \to A$ such that $\tau_d = \tau_{d'}$. Therefore if τ_d may be defined from a progressive block map d is the only possibility. Since $d(00) = 0 = d(01)$ d is not progressive. With a little work one can check that τ_d is a homeomorphism on $Z(0) \cup Z(1)$ and $Z(2) \cup Z(3)$ such that $\tau_d(Z(0) \cup Z(1)) = A^{\mathbb{N}} = \tau_d(Z(2) \cup Z(3))$. Therefore τ_d is a local homeomorphism.

Now we generalize the idea of a progressive block map by defining a weakly progressive block map. We prove that if d is weakly progressive, then τ_d is a local homeomorphism.

Definition 3.7. Fix $n, m \in \mathbb{N}$ and let a block map $d : A^n \to A$ have the property that for every $\mu \in A^n$ and every $\nu \in A^m$ such that $d(\mu) = \nu_1$ there exists a unique $a \in A$ such that $p_{d,m}^{\mu_1 \cdots \mu_{n-1}}(a\alpha) = d(\mu_1 \cdots \mu_{n-1}a)d(\mu_2 \cdots \mu_{n-1}a\alpha_1) \cdots = \nu$ has a solution $\alpha \in A^{m-1}$. Then we say that d is *weakly progressive* of order m.

Observe that d is progressive if and only if d is weakly progressive of order 1.

Example 3.8. Let $A = \{0, 1, 2, 3\}$ and define $d : A^2 \to A$ by:

$d(00) = 0$	$d(01) = 0$	$d(02) = 1$	$d(03) = 1$
$d(10) = 2$	$d(11) = 2$	$d(12) = 3$	$d(13) = 3$
$d(20) = 0$	$d(21) = 0$	$d(22) = 1$	$d(23) = 1$
$d(30) = 2$	$d(31) = 2$	$d(32) = 3$	$d(33) = 3.$

The function d is weakly progressive of order 2.

Remark 3.9. *The block map from Example 3.6 is weakly progressive of order 2.*

Proposition 3.10. *Let $d : A^n \to A$ be a block map and fix $x_1 \cdots x_{n-1} \in A^{n-1}$. If d is weakly progressive, then*

$$\tau_d : Z(x_1 \cdots x_{n-1}) \to \bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$$

is bijective.

Proof. Fix m such that d is weakly progressive of order m. Notice that for $x \in Z(x_1 \cdots x_{n-1})$, $\tau_d(x) \in Z(d(x_1 \cdots x_n)) \subseteq \bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$. Therefore we have $\tau_d(Z(x_1 \cdots x_{n-1})) \subseteq \bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$. Let $y \in Z(d(x_1 \cdots x_{n-1}a))$ for some $a \in A$. We want to show that there exists a unique $x \in Z(x_1 \cdots x_{n-1})$ such that $\tau_d(x) = y$. Notice that $x_1 \cdots x_{n-1}a \in A^n$ and $y_1 \cdots y_m \in A^m$ satisfy $d(x_1 \cdots x_{n-1}a) = y_1$. So since d is weakly progressive there exists a unique $a_1 \in A$ such that

$$p_{d,m}^{x_1 \cdots x_{n-1}}(a_1\alpha) = d(x_1 \cdots x_{n-1}a_1)d(x_2 \cdots x_{n-1}a_1\alpha_1)\cdots = y_1 \cdots y_m$$

for some $\alpha \in A^{m-1}$. Now consider $x_2 \cdots x_{n-1}a_1\alpha_1 \in A^n$ and $y_2 \cdots y_{m+1} \in A^m$ such that $d(x_2 \cdots x_{n-1}a_1\alpha_1) = y_2$. Since d is weakly progressive there exists a unique $a_2 \in A$ such that

$$p_{d,m}^{x_1 \cdots x_{n-1}}(a_2\beta) = d(x_2 \cdots x_{n-1}a_1a_2)d(x_3 \cdots x_{n-1}a_1a_2\beta_1) = y_2 \cdots y_{m+1}$$

for some $\beta \in A^{m-1}$. We may continue in this manner to construct $x = x_1 \cdots x_{n-1}a_1a_2 \cdots$ such that $\tau_d(x) = y$, hence the function is surjective. Since each a_i was unique we have $\tau_d|(Z(x_1 \cdots x_{n-1}))$ injective. $\square$

Observe that if d is progressive, then $\bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a)) = A^{\mathbb{N}}$. Thus τ_d is $|A^{n-1}|$ to 1.

Theorem 3.11. *If $d : A^n \to A$ is weakly progressive block map then the induced sliding block code $\tau_d : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a local homeomorphism.*

Proof. By Proposition 3.10 $\tau_d(Z(x_1 \cdots x_{n-1})) = \bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$ so $\tau_d(Z(x_1 \cdots x_{n-1}))$ is open. Since τ_d is continuous, $\tau_d|_{Z(x_1 \cdots x_{n-1})}$ is also continuous. The set $Z(x_1 \cdots x_{n-1})$ is compact since it is a cylinder set. Recall $A^{\mathbb{N}}$ is Hausdorff, hence $\bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a)) \subseteq A^{\mathbb{N}}$ is Hausdorff. So we have $\tau_d|_{Z(x_1 \cdots x_{n-1})}$ is a continuous bijective function from the compact space $Z(x_1 \cdots x_{n-1})$ to the Hausdorff space $\bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$. Therefore by [4, Theorem 5.8], $\tau_d|_{Z(x_1 \cdots x_{n-1})}$ is a homeomorphism. Hence τ_d is a local homeomorphism. $\square$

4. Continuous functions that *-commute with the Shift

The concept of *-commuting for functions was introduced in [1] and further examined in [2]. In this section we introduce the concept of a regressive block map (Definition 4.5) and prove that sliding block codes that *-commute with the shift map σ are exactly those defined from regressive block maps.

Definition 4.1. Let X be a set. Two functions $S, T : X \to X$ *-commute if they commute and given $(y, z) \in X \times X$ such that $S(y) = T(z)$ there exists a unique $x \in X$ such that $T(x) = y$ and $S(x) = z$.

$$S(y) = T(z)$$
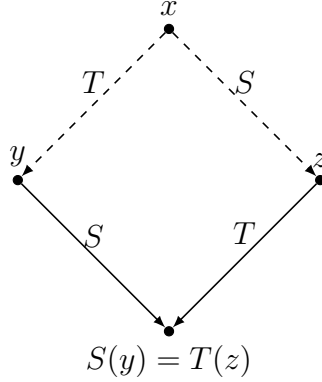
Remark 4.2. *Definition 4.1 is the same as the definition of "star-commuting" in* [2, Definition 10.1]. *This is equivalent to the definition of "∗-commuting" given in* [1, Definition 5.6].

Example 4.3. Let $A = \{0, 1\}$ and define the function $d : A \to A$ by $d(0) = 1$ and $d(1) = 0$. For $x \in A^{\mathbb{N}}$ denote $\tau_d(x) = \overline{x}$. By Lemma 2.3 we know that τ_d commutes with σ. Let $y, z \in A^{\mathbb{N}}$ be such that $\sigma(y) = \tau_d(z)$. Since τ_d is bijective, observe that $\overline{y}$ is the unique element in $A^{\mathbb{N}}$ such that $\tau_d(\overline{y}) = y$. We also have that $\tau_d(\sigma(\overline{y})) = \sigma(\tau_d(\overline{y})) = \sigma(y) = \tau_d(z)$ and since τ_d is bijective $\sigma(\overline{y}) = z$. So τ_d ∗-commutes with σ.

Example 4.4. Recall Example 3.3 where we define $d : A^2 \to A$ by $d(a_1 a_2) = a_2$ so that $\tau_d = \sigma$. Let $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $w \in A^{\mathbb{N}}$. Observe that $\sigma(a_1 w) = w = \sigma(a_2 w)$ and $a_1 w \neq a_2 w$. Therefore σ does not ∗-commute with itself.

In the two previous examples proving whether or not the function τ_d ∗-commutes with σ using the definition was not terribly difficult. However consider the following example: Let $A = \{0, 1, \cdots n - 1\}$ and define $d : A^n \to A$ by $d(a_1 \cdots a_n) = (a_1 + \cdots + a_n) \pmod{n}$. Determining whether or not the associated τ_d ∗-commutes with σ is extremely unpleasant. We would like to determine easily verifiable conditions on the block map d that determine when τ_d ∗-commutes with the shift.

Definition 4.5. The block map $d : A^n \to A$ is *regressive* if for each fixed $x_1 \cdots x_{n-1} \in A^{n-1}$ the function $r_d^{x_1 \cdots x_{n-1}} : A \to A$ defined by $r_d^{x_1 \cdots x_{n-1}}(a) = d(a x_1 \cdots x_{n-1})$ is bijective.

Example 4.6. Recall the block map from Example 3.6. Notice that when the second coordinate is fixed d is bijective. Therefore d is regressive.

Example 4.7. Recall the block map from Example 3.8. Since $d(00) = 0 = d(20)$, d is not regressive.

Example 4.8. For this example, all addition is modulo n. Let $A = \{0, 1, \cdots, n-1\}$ and define $d : A^n \to A$ by $d(a_1 \cdots a_n) = (a_1 + \cdots + a_n)$ (mod n). Fix $x_1 \cdots x_{n-1} \in A^{n-1}$ and let $x := x_1 + \cdots + x_{n-1}$. To see that r_d is injective, let $a_1, a_2 \in A$ and suppose $r_d^{x_1 \cdots x_{n-1}}(a_1) = r_d^{x_1 \cdots x_{n-1}}(a_2)$. Then

$$
\begin{aligned}
a_1 + x = d(a_1 x_1 \cdots x_{n-1}) &= r_d^{x_1 \cdots x_{n-1}}(a_1) \\
&= r_d^{x_1 \cdots x_{n-1}}(a_2) = d(a_2 x_1 \cdots x_{n-1}) = a_2 + x,
\end{aligned}
$$

therefore $a_1 = a_2$. Let $a \in A$. Then we have $r_d^{0_1 \cdots 0_{n-1}}(a) = a$. Therefore d is regressive.

Theorem 4.9. *The block map $d : A^n \to A$ is regressive if and only if the induced sliding block code $\tau_d : A^{\mathbb{N}} \to A^{\mathbb{N}}$ $*$-commutes with the shift map σ.*

Proof. By Lemma 2.3, τ_d commutes with σ. Suppose we have $y, z \in A^{\mathbb{N}}$ such that $\sigma(y) = \tau_d(z)$. Since d is regressive there exists a unique $x_1 \in A$ such that $r_d^{z_1 \cdots z_{n-1}}(x_1) = d(x_1 z_1 \cdots z_{n-1}) = y_1$. Notice that $y_{i+1} = \sigma(y)_i = \tau_d(z)_i = \tau_d(x_1 z)_{i+1}$. So we have

$$
\tau_d(x_1 z) = d(x_1 z_1 \cdots z_{n-1}) \tau_d(x_1 z)_2 \tau_d(x_1 z)_3 \cdots = y_1 y_2 y_3 \cdots = y
$$

and $\sigma(x_1 z) = z$. To see that $x_1 z$ is unique suppose there exists $w \in A^{\mathbb{N}}$ such that $\tau_d(w) = y$ and $\sigma(w) = z$. Then $w = az$ for some $a \in A$. Notice that $d(az_1 \cdots z_{n-1}) = \tau_d(az)_1 = y_1 = d(x_1 z_1 \cdots z_{n-1})$. Since d is regressive $a = x_1$. Therefore τ_d $*$-commutes with σ.

Conversely, fix $x_1 \cdots x_{n-1} \in A^{n-1}$. Suppose for $a_1, a_2 \in A$ we have $r_d^{x_1 \cdots x_{n-1}}(a_1) = r_d^{x_1 \cdots x_{n-1}}(a_2)$. Then let $z \in Z(x_1 \cdots x_{n-1})$ and observe that

$$
\begin{aligned}
\tau_d(a_1 z)_1 = d(a_1 z_1, \cdots z_{n-1}) &= r_d^{x_1 \cdots x_{n-1}}(a_1) \\
&= r_d^{x_1 \cdots x_{n-1}}(a_2) = d(a_2 z_1, \cdots z_{n-1}) = \tau_d(a_2 z)_1.
\end{aligned}
$$

For $i \geq 2$ we have $\tau_d(a_1 z)_i = \tau_d(z)_{i-1} = \tau_d(a_2 z)_i$. So $\tau_d(a_1 z) = \tau_d(a_2 z)$ and $\sigma(a_1 z) = z = \sigma(a_2 z)$. Since τ_d $*$-commutes with σ we have $a_1 z = a_2 z$. Therefore $r_d^{x_1 \cdots x_{n-1}}$ is injective. Now let $a \in A$. Suppose $z \in Z(x_1 \cdots x_{n-1})$ and define $w = \tau_d(z)$. Then $aw, z \in A^{\mathbb{N}}$ satisfy $\sigma(aw) = \tau_d(z)$. Since τ_d and σ $*$-commute there exists a unique $v \in A^{\mathbb{N}}$ such that $\sigma(v) = z$ and $\tau_d(v) = aw$. Since $\sigma(v) = z$, there exists $b \in A$ such that $v = bz$. So we have $a = \tau_d(v)_1 = \tau_d(bz)_1 = d(bz_1 \cdots z_{n-1}) = d(bx_1 \cdots x_{n-1})$. So $b \in A$ such that $r_d^{x_1 \cdots x_{n-1}}(b) = d(bx_1 \cdots x_{n-1}) = a$. Therefore d is regressive. $\square$

5. Local homeomorphisms that ∗-commute with the shift

In this section we examine properties of sliding block codes that are local homomorphisms and ∗-commute with the shift. In Theorem 5.14 we show that this class of functions is precisely the k-fold covering maps defined from regressive block maps.

Definition 5.1. Let X be a topological space, $\phi : X \to X$ be a function and $k \in \mathbb{N}$. We define the sets $Z_k^\phi := \{y \in X : |\phi^{-1}(y)| = k\}$ and $Z_{\geq k}^\phi := \{y \in X : |\phi^{-1}(y)| \geq k\}$.

Remark 5.2. Let $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ commute with σ and fix $y \in A^{\mathbb{N}}$. If there exists $x \in A^{\mathbb{N}}$ such that $\phi(x) = y$, then $\sigma(y) = \sigma(\phi(x)) = \phi(\sigma(x))$. This shows that if $\phi^{-1}(y) \neq \emptyset$, then $\phi^{-1}(\sigma(y)) \neq \emptyset$.

Definition 5.3. Let $X \subseteq A^{\mathbb{N}}$. We say that X is *shift invariant* if $\sigma(X) = X$.

Proposition 5.4. *If $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ ∗-commutes with the shift map σ, then Z_k^ϕ is shift invariant for all $k \in \mathbb{N}$.*

Proof. Let $y \in A^{\mathbb{N}}$ and fix $k \in \mathbb{N}$ such that $\sigma(y) \in Z_k^\phi$. Then $\sigma(y)$ has k preimages under ϕ and we define $\{z^i\}_{i=1}^k = \phi^{-1}(\sigma(y))$ (see Figure 1 below). Since σ and ϕ ∗-commute, for each z^i there exists a unique x^i such that $\phi(x^i) = y$ and $\sigma(x^i) = z^i$. So $|\phi^{-1}(y)| \geq k$, but we want to show that $|\phi^{-1}(y)| = k$. Suppose $x \in \phi^{-1}(y)$. Then $\phi(\sigma(x)) = \sigma(\phi(x)) = \sigma(y)$. So $\sigma(x) \in \phi^{-1}(\sigma(y)) = \{z^i\}_{i=1}^k$, and there exists i such that $\sigma(x) = z^i$. Hence $\phi(x) = y$, $\sigma(x) = z^i$, but x^i is the unique element with those properties thus $x = x^i$. Therefore $|\phi^{-1}(y)| = k$ that means $y \in Z_k^\phi$.
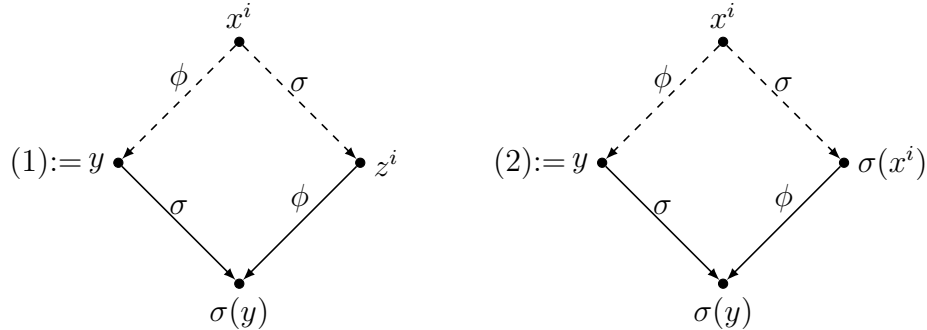
Conversely, let $y \in A^{\mathbb{N}}$ and fix $k \in \mathbb{N}$ such that $y \in Z_k^\phi$. Then define $\{x^i\}_{i=1}^k = \phi^{-1}(y)$ (see Figure 2 above). Suppose $w \in \phi^{-1}(\sigma(y))$ (which exists by Remark 5.2). Since σ and ϕ ∗-commute there exists

x such that $\phi(x) = y$ and $\sigma(x) = w$. However $\{x^i\}_{i=1}^k = \phi^{-1}(y)$ so $x = x^i$ for some i. So for each $w \in \phi^{-1}(\sigma(y))$, $w = \sigma(x^i)$ for some i. So $|\phi^{-1}(\sigma(y))| \leq k$. Suppose $|\phi^{-1}(\sigma(y))| < k$, then there exists $x^i, x^j \in \phi^{-1}(y)$ with $i \neq j$ such that $\sigma(x^i) = \sigma(x^j) = z$, say. So we have $y, z \in A^{\mathbb{N}}$ such that $\sigma(y) = \phi(z)$ and $x^i \neq x^j$ such that $\phi(x^i) = y = \phi(x^j)$ and $\sigma(x^i) = z = \sigma(x^j)$. Since ϕ and σ $*$-commute $x^i = x^j$, that is a contradiction. Therefore $|\phi^{-1}(\sigma(y))| = k$ which means $\sigma(y) \in Z_k^\phi$. □

Proposition 5.5. *If $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a local homeomorphism, then $Z_{\geq k}^\phi$ is open in $A^{\mathbb{N}}$ for all $k \in \mathbb{N}$.*

Proof. Let $y \in A^{\mathbb{N}}$ and fix $k \in \mathbb{N}$ such that $y \in Z_{\geq k}^\phi$. Then there exists $l \geq k$ such that $\phi^{-1}(y) = \{x_i\}_{i=1}^l$ where each x_i is distinct. Therefore $y \in Z_l^\phi$. Since ϕ is a local homeomorphism, for each x_i there exists a neighborhood U_i containing x_i such that x_j is not an element of U_i for $i \neq j$, $y \in \phi(U_i)$ for each i, and $\phi(U_i)$ is open in $A^{\mathbb{N}}$. Since $A^{\mathbb{N}}$ is Hausdorff, let $x_i \in V_i$ for each i and $V_i \cap V_j = \emptyset$ for each $i \neq j$. Then define $W_i = V_i \cap U_i$. Thus the set $\{W_i\}_{i=1}^l$ are pairwise disjoint. Now let $W = \bigcap_{i=1}^l \phi(W_i)$. Then W is open in $A^{\mathbb{N}}$ and $y \in W$. We want to show that $W \subseteq Z_{\geq k}^\phi$. Let $z \in W$. Then $z \in \phi(W_i)$ for each i, so there exists $w_i \in W_i$ such that $\phi(w_i) = z$. Since $\{W_i\}_{i=1}^l$ are pairwise disjoint, each w_i is distinct. Therefore z has at least l preimages. Hence $z \in Z_{\geq l}^\phi \subseteq Z_{\geq k}^\phi$, thus $W \subseteq Z_{\geq k}^\phi$. Therefore $Z_{\geq k}^\phi$ is open. □

Remark 5.6. *It is important to note that the only shift invariant open sets in $A^{\mathbb{N}}$ are $\emptyset$ and $A^{\mathbb{N}}$. Any non-empty open set U contains a basic open set $Z(x_1, \cdots, x_l)$ for some $l \in \mathbb{N}$. If $\sigma(U) = U$, then $\sigma^k(U) = U$ for any $k \in \mathbb{N}$. So $\sigma^l(Z(x_1, \cdots, x_l)) = A^{\mathbb{N}} \subseteq U$.*

Lemma 5.7. *If $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a local homeomorphism, then there exists an $M \in \mathbb{N}$ such that $\{Z(\mu) : \mu \in A^M\}$ is a finite covering of $A^{\mathbb{N}}$ by disjoint sets and ϕ is a homeomorphism on each $Z(\mu)$.*

Proof. Let $\{W_\alpha\}$ be a covering basic open sets of $A^{\mathbb{N}}$ such that ϕ is a homeomorphism on each set and let $\{W_i\}_{i=1}^n$ be a finite subcover. By Remark 2.4 there exists $M \in \mathbb{N}$ and $T \subseteq A^M$ such that $\bigcup_{i=1}^n W_i = \bigsqcup_{\nu \in T} Z(\nu)$. Since $\{W_i\}_{i=1}^n$ is a cover, $T = A^M$. Observe that for each $\nu \in A^M$, $Z(\nu) \subseteq W_i$, therefore ϕ is a homeomorphism on each $Z(\nu)$. □

Proposition 5.8. *If the sliding block code $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a local homeomorphism that $*$-commutes with the shift map σ, then ϕ is surjective and there exists $k \in \mathbb{N}$ such that ϕ is k-to-1.*

Proof. Since ϕ is a local homeomorphism, by Lemma 5.7 there exists an $M \in \mathbb{N}$ such that ϕ is a homeomorphism on each $Z(\mu)$ for all $\nu \in A^M$. Observe that for any $l \in \mathbb{N}$ such that $l > |A^M|$ we have $Z_l^{\phi} = \emptyset$. Let $k := \max\{l : Z_l^{\phi} \neq \emptyset\}$ and notice that $Z_k^{\phi} = Z_{\geq k}^{\phi}$. Since ϕ ∗-commutes with σ, by Proposition 5.4 we have $\sigma(Z_k^{\phi}) = Z_k^{\phi}$. By Proposition 5.5 $Z_k^{\phi} = Z_{\geq k}^{\phi}$ is open and by Remark 5.6 $Z_k^{\phi} = A^{\mathbb{N}}$. Thus every element of $A^{\mathbb{N}}$ has exactly k preimages under ϕ.

Suppose ϕ is not surjective. Then there exists $y \in A^{\mathbb{N}}$ such that $y \in Z_0$. We have just proven that every element of $A^{\mathbb{N}}$ must have the same number of preimages under ϕ. Then $k = 0$ which means ϕ is not defined for any element in $A^{\mathbb{N}}$. Thus ϕ must be surjective. $\qquad\square$

Definition 5.9. Let $p : E \to B$ be a continuous surjective function. The open set U of B is said to be *evenly covered* by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_{α} in E such that for each α, the restriction of p to V_{α} is a homeomorphism of V_{α} onto U.

Definition 5.10. Let $p : E \to B$ be a continuous surjective function. If every point $b \in B$ has a neighborhood U that is evenly covered by p, then p is called a *covering map* and E is said to be a *covering space* of B. If $p^{-1}(b)$ has k elements for every $b \in B$, then E is called a *k-fold covering* of B. The condition that p be a local homeomorphism does not suffice to ensure that p is a covering map (see [7, Chapter 9, page 338, Example 2]).

Example 5.11. The shift σ is a $|A|$-fold covering map.

Example 5.12. Define $V := Z(0) \cup Z(1)$ and $W := Z(2) \cup Z(3)$. Observe that for both the τ_d functions from Example 3.6 and Example 3.8 we have $\tau_d(V) = A^{\mathbb{N}} = \tau_d(W)$. Therefore the τ_d functions are 2-fold covering maps.

Proposition 5.13. *If the sliding block code $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a local homeomorphism that ∗-commutes with the shift map σ, then ϕ is a k-fold covering map.*

Proof. The function ϕ is continuous by definition and surjective by Proposition 5.8. Let $y \in A^{\mathbb{N}}$, then $\phi^{-1}(y) = \{x_i\}_{i=1}^{k}$ for some $k \in \mathbb{N}$ by Proposition 5.8. Since ϕ is a local homeomorphism there exists an open neighborhood W_i of x_i such that $\phi : W_i \to \phi(W_i)$ is a homeomorphism and $\phi(W_i)$ for $1 \leq i \leq k$. Since $A^{\mathbb{N}}$ is Hausdorff we may define open sets W_i' such that $x_i \in W_i' \subseteq W_i$ and $\{W_i'\}_{i=1}^{k}$ are pointwise disjoint. Notice that $\phi|_{W_i'}$ is a homeomorphism onto its image and $\phi(W_i')$ is

open. Let $U := \cap_{i=1}^{k} \phi(W_i')$. Then U is an open set such that $y \in U$. Define $V_i := \phi^{-1}(U) \cap W_i'$. Then $x_i \in V_i$ and the V_i's are open and pairwise disjoint. Notice that $\phi^{-1}(U) = \bigsqcup_{i=1}^{k} V_i$. Observe that $\phi|_{V_i}$ is a homeomorphism and $\phi(V_i) = \phi(\phi^{-1}(U) \cap W_i') = U \cap \phi(W_i') = U$. Hence $\phi(V_i)$ is onto U for each i. Therefore $A^{\mathbb{N}}$ is evenly covered by ϕ. By Theorem 5.8 ϕ is k-to-1 for some $k \in \mathbb{N}$, therefore ϕ is a k-fold covering map. $\qquad\square$

Theorem 5.14. *A sliding block code $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a local homeomorphism and $*$-commutes with the shift map σ if and only if ϕ is a k-fold covering map defined from a regressive block map.*

Proof. Since ϕ is a sliding block code that $*$-commutes with the shift, by Theorem 4.9 there exists a regressive block map d such that $\tau_d = \phi$. Since ϕ is a local homeomorphism that $*$-commutes with the shift, by Proposition 5.13 ϕ is a k-fold covering map.

Conversely, if ϕ is a k-fold covering map, then ϕ is a local homeomorphism by definition. Since ϕ is defined from a regressive block map, by Theorem 4.9 ϕ $*$-commutes with the shift. $\qquad\square$

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